Kinetic Equations

Text of the Exercises

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Exercise 1

Let f be a continuous function on $\mathbb{R}$ (or $\mathbb{R}_+$). Assume in addition that $\lim_{|x|\to+\infty} f(x) = 0$. Prove that f is uniformly continuous on $\mathbb{R}$ (or $\mathbb{R}_+$).

Hint: One may rely on the Heine-Cantor theorem, stating that any continuous function f defined between two metric spaces E and F is uniformly continuous if E is a compact set.

Exercise 2

Let $r < r'$ be two positive real numbers. Define the function:

$$\Delta_t\left(r, r'\right) = f(t, r) - f\left(t, r'\right), \tag{1}$$

where f is a solution of the Boltzmann equation:

$$\partial_t f + L(f) f = J(f). \tag{2}$$

Assume that f satisfies

$$0 \leqslant f(t, r) \leqslant \frac{c}{(1+r)^k} \tag{3}$$

for all $(t, r) \in \mathbb{R}_+^2$, with $k > 6$, and where

$$J(f)(t, r) = 4\int_0^{+\infty}\int_0^{+\infty} f(t, u) f(t, v) G(r, u, v)\, uv du dv, \tag{4}$$

$$G(r, u, v) = \begin{cases} 0 & \text{if } u^2 + v^2 \leqslant r^2, \\ 1 & \text{if } u \geqslant r,\ v \geqslant r, \\ \frac{v}{r} & \text{if } u \geqslant r,\ v \leqslant r, \\ \frac{u}{r} & \text{if } u \leqslant r,\ v \geqslant r, \\ \frac{\sqrt{u^2+v^2-r^2}}{r} & \text{if } u^2+v^2 \geqslant r^2,\ u \leqslant r,\ v \leqslant r, \end{cases} \tag{5}$$

$$L(f)(t, r) = \int_0^r \left(2r + \frac{2u^2}{3r}\right) f(t, u)\, u^2 du + \int_r^{+\infty} \left(2u + \frac{2r^2}{3u}\right) f(t, u)\, u^2 du. \tag{6}$$

(i) Prove that Δ_t solves the differential equation:

$$\partial_t \Delta_t\left(r, r'\right) + L(f)(t, r)\, \Delta_t\left(r, r'\right) = \tag{7}$$

$$= J(f)(t, r) - J(f)\left(t, r'\right) + f\left(t, r'\right)\Big(L(f)\left(t, r'\right) - L(f)(t, r)\Big). \tag{8}$$

(ii) Prove that there exists a positive number $\rho = \rho(t, r, r') \in (r, r')$ such that (7) can be rewritten as

$$\partial_t \Delta_t (r, r') + L(f)(t, r)\, \Delta_t (r, r') = \tag{9}$$

$$= (r' - r) \left(f(t, r')\, \partial_r L(f)(t, \rho) - \partial_r J(f)(t, \rho) \right). \tag{10}$$

Hint: Introduce a suitable function in order to apply the Mean Value Theorem.

(iii) Proof (and justify carefully) that

$$\partial_r L(f)(t, r) = \int_0^r \left(2 - \frac{2u^2}{3r^2} \right) u^2 f(t, u)\, du + \frac{4r}{3} \int_0^{+\infty} u f(t, u)\, du \tag{11}$$

and

$$\partial_r J(f)(t, r) = -\frac{8}{r^2} \left(\int_0^r u^2 f(t, u)\, du \right) \left(\int_r^{+\infty} v f(t, v)\, dv \right) \tag{12}$$

$$- \frac{4}{r^2} \int_0^r u f(t, u) \int_{\sqrt{r^2 - u^2}}^r \frac{u^2 + v^2}{\sqrt{u^2 + v^2 - r^2}} v f(t, v)\, dv du. \tag{13}$$

(iv) Prove that the absolute values of the partial derivatives $|\partial_r L(f)|$ and $|\partial_r J(f)|$ can be bounded respectively by two constants $M_1, M_2 > 0$ that do not depend on t and r.

Hint: To control the second term of $\partial_r J(f)$, one can prove and use that

$$f(u) f(v) \leqslant \frac{c^2}{\left(1 + \frac{r}{\sqrt{2}} \right)^k}. \tag{14}$$

in the domain of the second integral.

(v) Using the uniform controls on $|\partial_r L(f)|$ and $|\partial_r J(f)|$, prove that for all $(t, r) \in \mathbb{R}_+^2$:

$$\left| \Delta_t (r, r') \right| \leqslant e^{-\int_0^t L(f)(\tau, r) d\tau} \left| \Delta_0 (r, r') \right| \tag{15}$$

$$+ \int_0^t e^{-\int_s^t L(f)(\tau, r) d\tau} \left| r' - r \right| (M_1 c + M_2)\, ds. \tag{16}$$

(vi) Taking for granted that there exists a constant $k > 0$ such that

$$L(f)(t, r) \geqslant k \tag{17}$$

for all $(t, r) \in \mathbb{R}_+^2$, deduce from (15) that for all $(t, r) \in \mathbb{R}_+^2$ we have:

$$\left| \Delta_t (r, r') \right| \leqslant \left| f_0(t, r) - f_0(t, r') \right| + \left| r' - r \right| \frac{M_1 c + M_2}{k}. \tag{18}$$

(vii) Use the previous point to prove the uniform continuity in r (uniformly in t) of the solutions of the Boltzmann equation.

Hint: Use the result of the first exercise.